SOME PROPERTIES OF CLASSICAL POLYNOMIALS AND THEIR APPLICATION TO CONTACT PROBLEMS

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Some new properties are established for Jacobi polynomials (and, incidentally, also for those of Gegenbauer, Legendre and Chebyshev) which are then used for solving an integral equation that represents a number of plane contact problems (with contact along an interval) and also three-dimensional contact problems with a circular region of contact.

1. Suppose that the linear integral operator L transforms

$$x^m \rho(x) \qquad (m = 0, 1, 2 \ldots)$$

where $\rho(x)$ is an integrable function, into an nth degree polynomial multiplied by $\rho(x)$, namely

$$L[x^m \rho(x)] = \rho(x) \sum_{k=0}^{m} a_k^{(m)} x^k \quad (m = 0, 1, 2 \ldots) \tag{1.1}$$

whereby

$$a_m^{(m)} \neq a_k^{(k)} \neq 0 \qquad (m, k = 0, 1, 2 \ldots) \tag{1.2}$$

Under this hypothesis, one can construct a family of polynomials

$$p_m(x) = \sum_{j=0}^{m} c_j^{(m)} x^j, \quad c_m^{(m)} = 1 \qquad (m = 0, 1, 2 \ldots) \tag{1.3}$$

such that

$$L[p_m(x) \rho(x)] = \mu_m \rho(x) p_m(x), \qquad \mu_m = a_m^{(m)} \quad (m = 0, 1, 2 \ldots) \tag{1.4}$$

Indeed, comparing (1.4) with (1.1) and taking into account (1.3), we see that

$$\sum_{j=0}^{m} c_j^{(m)} \sum_{k=0}^{j} a_k^{(j)} x^k = \mu_m \sum_{j=0}^{m} c_j^{(m)} x^j$$

Changing the order of summation on the left-hand side of this equation, and equating coefficients of like powers of x, we obtain the second formula of (1.4) and also the system of equations

$$a_m^{(m)} c_k^{(m)} = \sum_{j=k}^{m-1} a_k^{(j)} c_j^{(m)} + a_k^{(m)} \qquad (k = 0, 1, 2 \ldots, m-1) \tag{1.5}$$

By means of these equations we can determine the coefficients of the polynomial (1.3), since the determinant of this system is different from zero because of (1.2).

It should be noted that the family of polynomials mentioned will also exist when condition (1.2) is violated if the operator L is symmetric.

Starting with system (1.5), one can obtain a formula for the determination of the coefficients $c_k^{(m)}$. Omitting the details, we give the final result as

$$c_k^{(m)} = \frac{1}{\mu_m - \mu_k} \left(\sum_{j=0}^{m-k-2} A_{k,j}^{(m)} + \frac{a_{m-1}^{(m)} A_{k,m-k-2}^{m,m-1}}{\mu_m - \mu_{m-1}} \right)$$

$$(k = 0, 1, 2 \ldots m-2; \quad c_{m-1}^{(m)} = (\mu_m - \mu_{m-1})^{-1} a_{m-1}^{(m)}) \tag{1.6}$$

where

$$A_{n,j}^{(m)} = \sum_{r=n+1}^{m-1} \frac{a_n^{(r)} A_{r,j-1}^{(m)}}{\mu_m - \mu_r}, \qquad A_{n,j}^{m,s} = \sum_{r=n+1}^{s-j} \frac{a_n^{(r)} A_{r,j-1}^{m,s}}{\mu_m - \mu_r}, \qquad \begin{matrix} A_{n,0}^{(m)} = a_n^{(m)} \\ A_{r,0}^{m,s} = a_r^{(s)} \end{matrix}$$

2. Let us consider the integral operator

$$K_\mu^\gamma [\varphi(x)] = \int_0^a K_{\mu,\gamma}^{\nu,\lambda}(x, y)\, \varphi(y)\, dy, \quad K_{\mu,\gamma}^{\nu,\lambda}(x, y) = \frac{x_\lambda}{y^{\varepsilon+\lambda-1}} W_{\mu,\gamma}^{\nu}(x, y) \tag{2.1}$$

where

$$W_{\mu,\gamma}^{\nu}(x, y) = \int_0^\infty s^\nu J_\mu(sx) J_\gamma(sy)\, ds, \quad W_{\mu,\mu}^{\nu} = W_\mu^{\nu}; \tag{2.2}$$

($J_\mu(z)$ is a Bessel function)

The restrictions which have to be imposed on (2.2) will be given later.

Expressing the kernel of the operator (2.1) in the form

$$K_{\mu,\gamma}^{\nu,\lambda}(x, y) = \frac{k(x/y)}{y^{\nu+\varepsilon}}, \qquad k(z) = z^{\lambda} \int_0^{\infty} s^{\nu} J_{\mu}(sz) J_{\gamma}(s)\, ds \tag{2.3}$$

and setting $x = ae^{-\xi}$ and $y = ae^{-\eta}$, we obtain

$$K_{\mu}^{\gamma}[\varphi] = \left(\frac{e^{\xi}}{a}\right)^{\nu+\varepsilon} \int_0^{\infty} l(\xi - \eta)\, ae^{-\eta} \varphi(ae^{-\eta})\, d\eta, \qquad l(t) = \frac{k(e^{-t})}{e^{(\nu+\varepsilon)t}} \tag{2.4}$$

Let us now make use of the Fourier transformations

$$L(u) = \int_{-\infty}^{\infty} l(t)\, e^{iut} dt, \qquad \Phi(u) = \int_0^{\infty} ae^{-\eta} \varphi(ae^{-\eta})\, e^{i\eta u} d\eta \tag{2.5}$$

Here it is assumed that $\varphi(y) \equiv 0$ when $y > a$. On the basis of the convolution theorem for Fourier transforms, we have

$$K_{\mu}^{\gamma}[\varphi] = \left(\frac{e^{\xi}}{a}\right)^{\nu+\varepsilon} \frac{1}{2\pi} \int_{-\infty}^{\infty} L(u)\, \Phi(u)\, e^{iu\xi} du \tag{2.6}$$

Evaluating the first integral in (2.5) by the same method that was used in [1], we find that

$$L(u) = \frac{\Gamma(1/2[\mu + \nu + \lambda + \varepsilon - iu])\, \Gamma(1/2[1 + \gamma - \lambda - \varepsilon + iu])}{2^{1-\nu}\Gamma(1 + 1/2[\mu - \nu - \lambda - \varepsilon + iu])\, \Gamma(1/2[1 + \gamma + \lambda + \varepsilon - iu])} \tag{2.7}$$

If one assumes that

$$\varphi(x) = x^{2m+\mu+\lambda+\varepsilon}(a^2 - x^2)^{-\omega} = \varphi_m(x) \qquad (\omega = 1/2 - 1/2\,\nu,\ m = 0, 1, 2, \ldots) \tag{2.8}$$

then it is not difficult to evaluate the second integral of (2.5)

$$\Phi(u) = \frac{\Gamma(1 - \omega)\, \Gamma(m + 1/2[1 + \mu + \lambda + \varepsilon - iu])}{2a^{-2m-\mu-\nu-\lambda-\varepsilon}\, \Gamma(1 + m + 1/2[\mu + \nu + \lambda + \varepsilon - iu])}$$

Taking into consideration this last formula and also (2.7) and (2.6) with $\mu = \gamma$, we obtain the next formula

$$K_{\mu}^{\mu}[\varphi_m] = \frac{e^{(\nu+\varepsilon)\xi}\Gamma(1 - \omega)}{2^{2-\nu}a^{-2m-\mu-\lambda}} \frac{1}{2\pi} \times$$

$$\times \int_{-\infty}^{\infty} \frac{\Gamma(1/2[1 + \mu - \lambda - \varepsilon + iu])\, (1/2[1 + \mu + \lambda + \varepsilon - iu])_m\, e^{-iu\xi}}{\Gamma(1 + 1/2[\mu - \lambda - \varepsilon + iu])\, (1/2[\mu + \nu + \lambda + \varepsilon - iu])_{m+1}}\, du$$

$$(a)_n = \Gamma^{-1}(a)\, \Gamma(a + n) = a(a + 1)(a + 2)\ldots(a + n - 1)$$

The last integral is equal to the sum of the residues and hence

$$K_{\mu}{}^{\mu}[\varphi_m(x)] = a^{2m+\mu+\lambda}\sum_{k=0}^{m} a_k^{(m)} e^{-(2k+\mu+\lambda)\xi} \qquad \left(e^{-\xi} = \frac{x}{a}\right) \tag{2.9}$$

$$a_k{}^{(m)} = \frac{(\omega-k)_m\,\Gamma(1-\omega)\,\Gamma(1+k-\omega+\mu)}{(-1)^k 2^{1-\nu} k!\,(m-k)!\,\Gamma(1+k+\mu)} \tag{2.10}$$

Finally, setting $\varepsilon = 1-2\lambda$, $x = at$, $y = a\tau$, and taking into consideration (2.1) and (2.8) in place of (2.9), we derive

$$L^*[\rho(t)\,t^{2m}] = \rho(t)\sum_{k=0}^{m} a_k{}^{(m)} t^{2k} \qquad (\rho(t) = t^{\mu+\lambda}) \tag{2.11}$$

where

$$L^*\psi = a^{1+\nu}\int_0^1 \frac{(t\tau)^{\lambda} W_{\mu}{}^{\nu}(at, a\tau)\,\psi(\tau)\,d\tau}{\tau^{2\lambda-1}(1-\tau^2)^{\omega}} \qquad \left(\omega = \frac{1-\nu}{2}\right) \tag{2.12}$$

The performed operations that led us to relation (2.11), may be justified if one assumes* that $0 \leqslant \nu < 1$, $-1 < \mu = \gamma < \infty$.

In accordance with Section 1, the characteristic numbers of the operator (2.12) are given by the formula

$$\mu_m = \frac{(-1)^m 2^{\nu-1}\pi\Gamma(1+m+\mu-\omega)}{\sin\pi\omega\, m!\,\Gamma(1+m+\mu)\,\Gamma(\omega-m)} \qquad (m = 0, 1, 2 \ldots) \tag{2.13}$$

obtained from (2.10), and its characteristic functions ψ_m will have the form

$$\psi_m(t) = t^{\mu+\lambda} p_m(t^2) \tag{2.14}$$

Here the coefficients $p_m(x)$, given by the formula (1.3), can be determined with the aid of formulas (1.6) and (2.10).

However, in our case this determination is not necessary, since the indicated polynomials are related quite simply to the Jacobi polynomials $P_m^{(\alpha,\beta)}(x)$. To show this, we note first of all that the operator (2.12) can be transformed into a symmetric one by a well-known method [3, p.111] The characteristic functions of the symmetric operator will be

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* The referee of this work, N.A. Rostovtsev, pointed out that relation (2.11) can be obtained from his theorem on an elliptic stamp [2].

orthogonal to each other, which in our case is equivalent to the fulfillment of the requirements

$$\int_0^1 \frac{t^{1+2\mu}}{(1-t^2)^\omega} p_m(t^2)\, p_n(t^2)\, dt = 0 \qquad (m \neq n) \tag{2.15}$$

The change of variables

$$2t^2 = 1 - x,\ p_m^*(x) = p_m(^1/_2 - {}^1/_2\, x)$$

transforms (2.15) into

$$\int_{-1}^1 \frac{(1-x)^\mu}{(1+x)^\omega} p_m^*(x)\, p_n^*(x)\, dx = 0 \qquad (m \neq n)$$

These orthogonality conditions are, however, satisfied by the Jacobi polynomials $P_m^{(\mu,-\omega)}(x)$. Hence, on the basis of a well-known theorem [4, p.1037], we can conclude that

$$p_m^*(x) = B_m P_m^{(\mu,-\omega)}(x), \qquad p_m(t^2) = B_m P_m^{(\mu,-\omega)}(1-2t^2)\ (B_m = \text{const}) \tag{2.16}$$

Taking into account (2.11) to (2.16), we now arrive at the fundamental conclusion that

$$L^*[t^{\mu+\lambda} P_m^\mu(t)] = \mu_m t^{\mu+\lambda} P_m^\mu(t) \qquad (0 \leqslant t \leqslant 1) \tag{2.17}$$

where

$$P_m^\mu(t) = P_m^{(\mu,-\omega)}(1-2t^2) \tag{2.18}$$

3. Let us now consider the more interesting particular cases of relation (2.17), and also some of their consequences.

We begin with the cases when $\mu = \nu = \lambda = 0$; noting first, that on the basis of formulas 8.962 (1) and 8.911 (2) of [4], we have the following relations:

$$P_m^{(0,-1/2)}(1-2x^2) = P_{2m}(\sqrt{1-x^2}) \quad (P_m(z) \text{ is a Legendre polynomial}) \tag{3.1}$$

From (2.17) we obtain with the aid of (2.12), (2.13) and (3.1), the equation

$$a\int_0^1 W_0^\circ(at,\ a\tau)\, \frac{\tau P_{2m}(\sqrt{1-\tau^2})}{\sqrt{1-\tau^2}}\, d\tau = \frac{\pi}{2}\ \frac{(2m-1)!!^2}{2m!!^2} P_{2m}(\sqrt{1-t^2}) \tag{3.2}$$

which agrees with the results of [5], where a direct proof of relation (3.2) was given.

The case $\mu = \pm 1/2$, $\lambda = 1/2$. First of all we note that

$$\pi \sqrt{xy} W_{\mp 1/2}^{\nu}(x, y) = \Gamma(\nu) \cos {}^1/_2 \nu\pi [| x - y |^{-\nu} \pm (x + y)^{-\nu}] \qquad (3.3)$$

In order to prove this, one must take into account (2.2) and also formulas 3.762 and 8.463 of [4]. Let us also note that formulas 8.962 (1) and 8.932 (3) of [4] yield the following relations:

$$(3.4)$$

$$\sqrt{\pi}\,\Gamma(m + {}^1/_2\nu) P_m^{(-1/_2, -\omega)}(1 - 2t^2) = (-1)^m \Gamma({}^1/_2\nu)\, \Gamma(m + {}^1/_2)\, C_{2m}^{\nu/2}(t)$$

$$\sqrt{\pi}\,(1 + m + {}^1/_2\nu)\, t P_m^{(1/_2, -\omega)}(1 - 2t^2) = (-1)^m \Gamma({}^1/_2\nu)\, \Gamma(m + {}^3/_2)\, C_{2m+1}^{\nu/2}(t)$$

($C_m^{\alpha}(t)$ is a Gegenbauer polynomial)

Taking into account (2.12) and (2.13), let us set $\mu = \mp 1/2$, $\lambda = 1/2$ in (2.17). Then using (3.3) and (3.4), as well as certain well-known properties of the gamma functions, we obtain

$$\int_0^1 \left[\frac{1}{|x - y|^{\nu}} \pm \frac{1}{(x + y)^{\nu}}\right] \frac{C_{\pm}^{\nu/2}(y)}{\sqrt{(1 - y^2)^{1-\nu}}}\, dy = \frac{\pi\Gamma(\nu + n_{\pm})}{\cos {}^1/_2\,\nu\pi\Gamma(\nu)\,(n_{\pm})!} C_{\pm}^{\nu/2}(x) \quad (3.5)$$

$$(0 \leqslant x \leqslant 1,\ C_+^{\alpha} = C_{2m}^{\alpha},\ C_-^{\alpha} = C_{2m+1}^{\alpha},\ n_+ = 2m,\ n_- = 2m + 1,\ m = 0, 1, 2, \ldots)$$

As a direct consequence of this we find that

$$\int_{-1}^1 \frac{C_m^{\nu/2}(y)\, dy}{|x - y|^{\nu} \sqrt{(1 - y^2)^{1-\nu}}} = \frac{\pi\Gamma(m + \nu)}{\cos {}^1/_2\,\nu\pi\Gamma(\nu)\, m!} C_m^{\nu/2}(x) \qquad (|x| \leqslant 1) \quad (3.6)$$

From the left- and right-hand sides of (3.6) we next subtract the left- and right-hand sides, respectively, of the following equation [4, p.841]:

$$\frac{\Gamma({}^1/_2\nu)}{\nu} \int_{-1}^1 \frac{C_m^{\nu/2}(y)}{\sqrt{(1 - y^2)^{1-\nu}}}\, dy = \begin{cases} 2\sqrt{\pi}\,\Gamma({}^1/_2 + {}^1/_2\nu)\,\nu^{-2} & (m = 0) \\ 0 & (m = 1, 2, 3, \ldots) \end{cases}$$

After this we let $\nu \to 0$, and make use of [4, p.1044]

$$\lim_{\nu \to 0} \Gamma({}^1/_2\nu)\, C_m^{\nu/2}(x) = \frac{2}{m} T_m(x) \qquad (T_m(x) \text{ is a Chebyshev polynomial})$$

and also of

$$\ln \frac{1}{|x-y|} = \lim_{\nu\to 0} \left(\frac{1}{|x-y|^\nu} - 1 \right) \frac{1}{\nu} \tag{3.7}$$

As a result of the indicated operations we obtain the known relation [6]

$$\frac{1}{\pi} \int_{-1}^{1} \ln \frac{1}{|x-y|} \frac{T_m(y)}{\sqrt{1-y^2}} dy = \begin{cases} \ln 2T_0(y) & (m=0) \\ m^{-1} T_m(y) & (m=1,2,\ldots) \end{cases} \tag{3.8}$$

In the derivation of the consequences of relation (2.17) we shall make use of a known theorem [7, p.263] on the representation of a kernel of a symmetric operator as a series of products of orthonormal characteristic functions. After the operator (2.12) has been symmetrized by the procedure mentioned [3, p.111], its orthonormal characteristic functions will be given, on the basis of (2.14) to (2.17), by

$$t^{\mu+\lambda} \left[\frac{2(1+2m+\mu-\omega)m!\,\Gamma(1+m+\mu-\omega)}{\Gamma(1+m+\mu)\,\Gamma(1+m-\omega)(1-t^2)^\omega} t^{1-2\lambda} \right]^{1/2} P_m^{\mu}(t) \tag{3.9}$$

Here we have made use of formula 7.3911 from [4]. We thus have obtained from relation (2.17) and from the theorem mentioned, the following expansion:

$$\frac{a^{1+\nu} W_\mu^{\ \nu}(at, a\tau)}{2^{\nu-1}(t\tau)^\mu} = \sum_{m=0}^{\infty} \frac{\Gamma^2(1+m+\mu-\omega) P_m^{\ \mu}(t) P_m^{\ \mu}(\tau)}{\Gamma^2(1+m+\mu)(1+2m+\mu-\omega)^{-1}} \tag{3.10}$$

As particular examples of this, or as a consequence of relations (3.2), (3.5), (3.6) and (3.8), we obtain the following expansions:

$$a \int_0^\infty J_0(axs)\, J_0(ays)\, ds =$$

$$= \pi \sum_{m=0}^{\infty} \frac{(2m-1)!!^2 P_{2m}(\sqrt{1-x^2})\, P_{2m}(\sqrt{1-y^2})}{2m!!^2 (4m+1)^{-1}} \qquad (0 \leqslant x,\ y \leqslant 1)$$

$$\frac{1}{|x-y|^\nu} \pm \frac{1}{(x+y)^\nu} = \frac{2^\nu \Gamma^2(1/2\,\nu)}{\cos 1/2\,\nu\pi\Gamma(\nu)} \sum_{m=0}^{\infty} \frac{C_\pm^{\ \nu/2}(x)\, C_\pm^{\ \nu/2}(y)}{(n_\pm + 1/2\nu)^{-1}} \qquad (0 \leqslant x,\ y \leqslant 1)$$

$$(C_+^{\ \alpha} = C_{2m}^{\alpha},\quad C_-^{\ \alpha} = C_{2m+1}^{\ \alpha},\ n_+ = 2m,\ n_- = 2m+1)$$

$$\frac{1}{|x-y|^\nu} = \frac{2^{\nu-1}\Gamma^2(1/2\nu)}{\cos 1/2\,\nu\pi\Gamma(\nu)} \sum_{m=0}^{\infty} \frac{C_m^{\ \nu/2}(x)\, C_m^{\ \nu/2}(y)}{(m + 1/2\,\nu)^{-1}} \qquad (-1 \leqslant x,\ y \leqslant 1)$$

$$\ln \frac{1}{|x-y|} = \ln 2 + 2 \sum_{m=1}^{\infty} \frac{T_m(x)\, T_m(y)}{m} \qquad (-1 \leqslant x,\ y \leqslant 1)$$

We note that these series, as well as the series in (3.10), are convergent in the mean [7].

4. Let us now consider the integral equation

$$\int_0^a K_{\mu,\gamma}^{\nu,\lambda}(x, y)\,\varphi(y)\,dy = f(x) \qquad (0 \leqslant x \leqslant a) \tag{4.1}$$

Certain contact problems in the theory of elasticity can be reduced to this integral equation when $\gamma = \mu = n$, $\varepsilon = \lambda = 0$ (see [1,5,8]). These contact problems involve circular contact regions ($\nu = 0$) in the form of a usual half-space, or half-spaces with a variable (by the power law) modulus of elasticity ($\nu \neq 0$). Other contact problems covered by this equation belong to the nonlinear theory of plasticity (in their first approximation) with the same type of contact regions.

We shall show that the above-mentioned plane problems can also be reduced to the integral equation (4.1).

Arutiunian [9] has shown that the plane contact problem of the nonlinear theory of plasticity (in its first approximation) can be reduced to the integral equation

$$\int_{-a}^{a} \frac{1}{|x-y|^{\nu}}\, p^*(y)\,dy = f(x) \qquad (|x| \leqslant a) \tag{4.2}$$

L.A. Galin has reduced the contact problem of elasticity theory in a half-space with a modulus of elasticity that changes according to a power law to the same integral equation.

If one denotes by $p_+^*(x)$ the solution of equation (4.2) for the even right-hand side of $f_+(x)$, and by $p_-^*(x)$ the solution for the odd right-hand side $f_-(x)$, then it is obvious that the finding of the solution of equation (4.2) is equivalent to solving the following two equations:

$$\int_0^a \left[\frac{1}{|x-y|^{\nu}} \pm \frac{1}{(x+y)^{\nu}}\right] p_\pm^*(y)\,dy = f_\pm(x) \qquad (0 \leqslant x \leqslant a) \tag{4.3}$$

Taking into account (3.3) and (2.1), we can now conclude that the above-mentioned plane contact problems for the symmetric case can be reduced to the integral equation (4.1) with $\gamma = \mu = -1/2$, $\lambda = 1/2$, $\varepsilon = 0$, and for the skew-symmetric case to the same integral equation with $\gamma = \mu = 1/2$, $\lambda = 1/2$, $\varepsilon = 0$.

In regard to the plane contact problems for the ordinary half-spaces, which, as is well known [10], are equivalent to the following integral equations:

$$\int_0^a \left[\ln \frac{1}{|x-y|} \pm \ln \frac{1}{x+y}\right] p_{\pm}(y)\, dy = f_{\pm}(x) \qquad (0 \leqslant x \leqslant a) \tag{4.4}$$

it can be said that they also are covered by equation (4.1) since their solutions, in view of (3.7), can be obtained by the limit process from the solutions of equation (4.3).

Thus the application of relation (2.17), and its special cases, makes it possible to obtain the solutions of the above-mentioned contact problems in the form of power series in terms of classical polynomials. This type of solution will be used in Section 5.

Here, however, we shall construct the solution of the integral equation (4.1) in the form of quadratures. At the same time we shall obtain a general formula which will express the solutions of spatial contact problems with a circular contact region, as well as plane problems with an interval of contact.

First we note that equation (4.1) can be reduced to an integral equation of the Wiener-Hopf type of the first kind. Indeed, setting in (4.1)

$$x = ae^{-\xi}, \quad y = ae^{-\eta}, \quad \varphi(ae^{-\xi})\, ae^{-\xi} = \chi(\xi), \quad f(ae^{-\xi})\, a^{\nu+\varepsilon} e^{-(\nu+\varepsilon)\xi} = g(\xi) \tag{4.5}$$

and taking into account (2.3), we obtain

$$\int_0^\infty l(\xi - \eta)\, \chi(\eta)\, d\eta = g(\xi) \qquad (0 \leqslant \xi < \infty) \tag{4.6}$$

where $l(t)$ is given by the second formula of (2.4).

In order to obtain the solution of equation (4.6), it is sufficient to obtain one for the more simple equation [1]

$$\int_0^\infty l(\xi - \eta)\, \chi_\zeta(\eta)\, d\eta = e^{i\zeta\xi} \qquad (\xi,\ \operatorname{Im} \zeta \geqslant 0) \tag{4.7}$$

and then make use of the formulas

$$\chi(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(-\zeta)\, \chi_\zeta(\xi)\, d\zeta, \qquad G(u) = \int_0^\infty g(\xi)\, e^{i\xi u}\, d\xi \tag{4.8}$$

Utilizing the method described in [1] for the solution of equation (4.7) (compare also [5]), we are led to the equations

$$\varphi_\zeta(x) = \frac{1}{x}\chi_\zeta\left(\ln\frac{a}{x}\right) = \frac{\Psi_-(-\zeta)\,x^{\gamma+\lambda+\varepsilon}}{\Gamma(^1/_2[1+\mu-\gamma+\nu])}\left\{\frac{a^{-\gamma-\nu-\lambda-\varepsilon}}{\sqrt{(a^2-x^2)^{1-\mu+\gamma-\nu}}} + \right.$$
$$\left. + \frac{\mu+\nu+\lambda+\varepsilon+i\zeta}{a^{-i\zeta}}\int_x^a \frac{t^{-1-\mu-\nu-\lambda-\varepsilon-i\zeta}dt}{\sqrt{(t^2-x^2)^{1-\mu+\gamma-\nu}}}\right\} \qquad (0 \leqslant x \leqslant a) \qquad (4.9)$$
$$(\Psi_-(-\zeta) = 2^{1-\nu}\Gamma(1 + {}^1/_2[\mu-\nu-\lambda-\varepsilon-i\zeta])\,\Gamma^{-1}(^1/_2[1+\gamma-\lambda-\varepsilon-i\zeta]))$$

After obtaining the solution $\chi_\zeta(\xi)$ of equation (4.6) for a special right-hand side, one can derive the solution for the general case with the aid of formula (4.8). However, it is more convenient to use for this purpose the known result of Krein [11] (compare also [1]). In view of (2.1), equation (4.1) can be written in the form

$$\int_0^a W_{\mu,\gamma}^{\nu}(x, y)\,\varphi^*(y)\,dy = f^*(x) \qquad (0 \leqslant x \leqslant a) \qquad (4.10)$$

where

$$\varphi^*(x) = x^{1-\lambda-\varepsilon}\varphi(x), \qquad f^*(x) = x^{-\lambda}f(x)$$

It can be shown that the function $\varphi_\zeta(x)x^{1-\lambda-\varepsilon}$ is a solution of equation (4.10) when

$$f^*(x) = a^{i\zeta}x^{-\nu-\lambda-\varepsilon-i\zeta}$$

and the solution of the integral equation

$$\int_0^a W_{\mu,\gamma}^{\nu}(x, y)\,q_\mu^{\gamma}(y, a)\,dy = 1 \qquad (0 \leqslant x \leqslant a) \qquad (4.11)$$

is, therefore, given by the formula

$$q_\mu^{\nu}(y;\ a) = y^{1-\lambda-\varepsilon}\left[a^{-i\zeta}\varphi_\zeta(y)\right]_{\zeta=i(\nu+\lambda+\varepsilon)} \qquad (4.12)$$

Taking note of (2.2), it is not difficult to see that the solution of the integral equation which is the adjoint equation of (4.11), will be given by the same formulas (4.9) and (4.12). Hereby, however, one must interchange the parameters μ and γ.

Having obtained the solution $q_\mu^{\gamma}(y;\ a)$ of the integral equation (4.11), and the solution $q_\gamma^{\mu}(y;\ a)$ of the adjoint equation for the construction of the solution of (4.10) or (4.1) with arbitrary right-hand sides, one can make use of the formulas of Krein [11]. These computations yield (see [1]) the result*

$$\varphi(x)=\frac{2^{1-\nu}x^{\gamma+\lambda+\varepsilon}}{\Gamma(^1/_2[1+\mu-\gamma+\nu])\,\Gamma(^1/_2[1-\mu+\gamma+\nu])}\left\{\frac{\Phi(a)}{\sqrt{(a^2-x^2)^{1-\mu+\gamma-\nu}}}-\right. \tag{4.13}$$

$$\left.-\int_x^a\frac{\Phi'(u)\,du}{\sqrt{(u^2-x^2)^{1-\mu+\gamma-\nu}}}\right\},\qquad \Phi(a)=a^{-\mu-\gamma-\nu}\frac{d}{da}\int_0^a\frac{s^{1+\mu-\lambda}f(s)\,ds}{\sqrt{(a^2-s^2)^{1+\mu-\gamma-\nu}}}$$

The formal procedure for obtaining formula (4.13) can be justified if one assumes that $0\leqslant\nu<1$, $|\mu-\nu|<1+\nu$, and if one requires that the function $x^{1+\mu-\lambda}f(x)$ be continuous in $0\leqslant x\leqslant 1$.

Substituting the appropriate values of the parameters in the derived formula (4.13), one obtains the solutions of the contact problems mentioned at the beginning of this section. For example, in order to obtain the solution $p_+^*(x)$ of equation (4.2) with an even right-hand side, one must set $\gamma=\mu=-1/2$, $\lambda=1/2$ and $\varepsilon=0$ (taking note of (4.3) and (3.3)). This yields

$$p_+^*(x)=\frac{2^{1-\nu}\cos{}^1/_2\,\nu\pi\Gamma(\nu)}{\pi\Gamma^2(^1/_2+{}^1/_2\nu)}\left[\frac{\Phi(a)}{\sqrt{(a^2-x^2)^{1-\nu}}}-\int_x^a\frac{\Phi'(u)\,du}{\sqrt{(u^2-x^2)^{1-\nu}}}\right] \tag{4.14}$$

$$\Phi(a)=a^{1-\nu}\frac{d}{da}\int_0^a\frac{f_+(s)\,ds}{\sqrt{(a^2-s^2)^{1-\nu}}}$$

which coincides with the result obtained by Arutiunian [9] if one corrects the error in his result where the factor 1/2 was omitted in the formula for $M(a)$.

Let us also find the solution of the plane contact problem for the ordinary half-space, i.e. the solution of the integral equation (4.4). If one takes into account equation (3.7), one can show that

$$p_\pm(x)=\lim_{\nu\to 0}p_\nu^\pm(x) \tag{4.15}$$

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* The integral equation (4.10) with $\mu=\gamma$ has been solved also by N.I. Ahiezer and V.A. Shcherbin [12], and independently of them, by V.I. Mossakovskai and N.A. Rostovtsev [2] (with a kernel different from the one used here). The solution methods of these authors can be extended also the case when $\mu\neq\gamma$. The advantage of the method given here lies in the fact that it yields the solution of (4.10) or (4.1) even in the case when (2.2) contains products of functions of a more general type than $J_\mu(x)$, and also when the kernel is the sum of functions of the type (2.2).

where $p_\nu^{\pm}(x)$ satisfy the following equations:

$$\int_0^a \left[\frac{1}{|x-y|^\nu} \pm \frac{1}{(x+y)^\nu}\right] p_\nu^{\pm}(y)\,dy = \gamma_\nu^{\pm} + \nu f_{\pm}(x) \qquad (4.16)$$

$$\gamma_\nu^{+} = 2\int_0^a p_\nu^{+}(y)\,dy, \qquad \gamma_\nu^{-} = 0$$

The solutions of these equations ar', however, easily found by the use of the general formula (4.13). O.atting the details, we give here the final results

$$p_+(x) = \frac{2}{\pi^2}\left\{\left[\Phi_+(a) - \frac{1}{\ln {}^1/_2\, a}\int_0^a \frac{f_+(s)\,ds}{\sqrt{a^2-s^2}}\right]\frac{1}{\sqrt{a^2-x^2}} - \int_x^a \frac{\Phi_+'(u)\,du}{\sqrt{x^2-u^2}}\right\}$$

$$p_-(x) = \frac{2x}{\pi^2}\left[\frac{\Phi_-(a)}{\sqrt{a^2-x^2}} - \int_x^a \frac{\Phi_-'(u)\,du}{\sqrt{x^2-u^2}}\right] \qquad (4.17)$$

$$\Phi_{\pm}(a) = a^{\pm 1}\frac{d}{da}\int_0^a \sqrt{\frac{s^{1\mp 1}}{a^2-s^2}}\, f_{\pm}(s)\,ds$$

The obtained solution $p_+(x)$ for the symmetric case can easily be reduced to the form found by Rostovtsev [10], and by Krein [11,13].

5. Let us apply relation (2.17), obtained in Section 2, to contact problems with bases of general type introduced in [1,5,6]. In the last one of these works it is shown that the contact problem with a circular contact region can be reduced to the integral equation

$$\int_0^\alpha K_n(x, y)\, y p_n^*(y)\,dy = g_n^*(x) \qquad (0 \leqslant x \leqslant \alpha, \quad n = 0, 1, 2\ldots) \qquad (5.1)$$

where

$$K_\mu(x, y) = \int_0^\infty G(t)\, J_\mu(tx)\, J_\mu(ty)\,dt \qquad (5.2)$$

The function $G(t)$, whose form is determined by the type of the basis (contact region), has the asymptotic property

$$G(t) = t^\nu[1 + o(1)], \qquad t \to \infty \qquad (5.3)$$

On the other hand, the plane contact problem with a contact interval $(-a, a)$ can be reduced, in accordance with [6], to the integral equation

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha} v(x-y)\, p^*(y)\,dy = g^*(x) \quad (|x| \leqslant \alpha), \qquad v(t) = \int_0^\infty G(t)\cos t\tau\,\frac{d\tau}{\tau}$$

By separating the plane contact problem into the symmetric and skew-symmetric parts, we can reduce the last integral equation, by the method used in (4.2), to the form

$$\int_0^\alpha K_{\mp^1/_2}(x, y)\,\sqrt{xy}\,p_\pm^*(y)\,dy = g_\pm^*(x) \qquad (0 \leqslant x \leqslant \alpha)$$

Let us introduce the equation

$$\int_0^\alpha K_\mu(x, y)\,(xy)^\lambda\, y\varphi(y)\,dy = f(x) \qquad (0 \leqslant x \leqslant \alpha) \tag{5.4}$$

It is easily seen that it is general enough to include the contact problems with circular contacts

$$p_n^*(x) = [\varphi(x)]_{\lambda=0,\ \mu=n} \qquad (n = 0, 1, 2, \ldots)$$

as well as plane problems (with one contact area)

$$p_\pm^*(x) = [x\varphi(x)]_{\lambda=^1/_2,\ \mu=\mp^1/_2}$$

For the approximate solution of equation (5.4) we shall use the method of [5], which is based on the separation of the singular part from the kernel and on the approximation of the remaining continuous part by means of polynomials.

Taking into account the asymptotic property given in (5.3), we can express the kernel (5.2) in the form

$$K_\mu(x, y) \approx W_\mu^{\ \nu}(x, y) - (xy)^\mu \sum_{k=0}^N A_k M_k(x, y) \tag{5.5}$$

where the singular part is determined by formula (2.2), while the continuous part is approximated by the partial sum of a series of polynomials of the type

$$M_k(x, y) = \sum_{j=0}^k a_{kj} x^{2(k-j)} y^{2j} \qquad (k = 0, 1, 2 \ldots) \tag{5.6}$$

Substituting (5.5) into (5.4), and setting

$$x = \alpha\xi, \qquad y = \alpha\eta, \qquad \alpha^{1-\nu+2\lambda}\varphi(\alpha\xi) = \chi(\xi)$$

we obtain in place of (5.4) the following approximate equation:

$$\alpha^{1+\nu}\int_0^1 \Big[W_\mu^{\ \nu}(\alpha\xi, \alpha\eta) - \Big(\frac{\xi\eta}{\alpha^{-2}}\Big)^\mu \sum_{k=0}^N A_k M_k(\alpha\xi, \alpha\eta)\Big](\xi\eta)^\lambda \eta\chi(\eta)\,d\eta = f(\alpha\xi) \qquad (0 \leqslant \xi \leqslant 1) \tag{5.7}$$

whose solution we shall try to find in the form

$$\chi(\eta)=\sum_{m=0}^{\infty} Y_m \frac{\eta^{\mu-\lambda} P_m{}^{\mu}(\eta)}{(1-\eta^2)^{\omega}} \qquad \left(\omega=\frac{1-\nu}{2}\right) \tag{5.8}$$

In order to find the unknown coefficients Y_m we substitute (5.8) into (5.7) and make use of relation (2.17). Next, using the orthogonality of the Jacobi polynomials, we integrate each term of equation (5.7) with respect to ξ over the interval (0, 1) with the weight

$$\xi^{1+\mu-\lambda}(1-\xi^2)^{-\omega} P_l{}^{\mu}(\xi)$$

and derive

$$\lambda_l Y_l - \alpha^{1+\nu+2\mu} \sum_{m=0}^{N-l} Y_m \sum_{\max(m,\,l)}^{N} A_k \alpha^{2k} B_{mk}{}^{(l)} = f_l \qquad (l=0,1,2\ldots N) \tag{5.9}$$

$$Y_l=\lambda_l^{-1} f_l \qquad (N \leqslant l < \infty)$$

where

$$\lambda_l=\frac{2^{\nu-2}\Gamma^2(1+l-\omega)}{l!^2(1+2l+\mu-\omega)}, \qquad B_{mk}{}^{(l)}=\sum_{j=m}^{k-l} a_{kj} b_{k-j}{}^{(l)} b_j{}^{(m)}$$

$$b_n{}^{(k)}=\int_0^1 \frac{\xi^{1+2n+2\mu} P_k{}^{\mu}(\xi)}{(1-\xi^2)^{\omega}}\,d\xi=\begin{cases} 0 & (k>n) \\ \dfrac{(-1)^k n!\,\Gamma(1+n+\mu)\,\Gamma(1+k-\omega)}{k!\,(n-k)!\,2\Gamma(2+k+n+\mu-\omega)} & (k\leqslant n)\end{cases}$$

$$f_l=\int_0^1 \frac{\xi^{1+\mu-\lambda} f(\alpha\xi)}{(1-\xi^2)^{\omega}} P_l{}^{\mu}(\xi)\,d\xi \qquad (l=0,1,2\ldots)$$

In the evaluation of the integral which determines $b_n{}^{(k)}$ we have made the substitution $1-2\xi^2=x$, and then made use of formula 7.391 (4) of [4]. It is important to note that system (5.9) has a triangular matrix of coefficients.

In addition to finding the solution of equation (5.7) in the form of an infinite series (it will be a series only if the right-hand side is not a polynomial) one can find it also in the form of quadratures by using the same arguments as those used in Section 4 of [5].

It is helpful to note that if one uses the procedure described in that work for the approximation of the kernel (5.2) in the form (5.5), then one must set

$$a_{kj}=\frac{k!\,\Gamma(1+k+\mu)}{j!\,(k-j)!\,\Gamma(1+k-j+\mu)\,\Gamma(1+j+\mu)}, \qquad A_k=\frac{(-1)^k C_{2(k+\mu)}^{(\nu)}}{4^{k+\mu} k!\,\Gamma(1+k+\mu)}$$

$$C_r^{(\nu)} = \int_0^A [s^\nu - G(s)]\, s^r ds \quad (r = -1, 0, 1, 2 \ldots), \qquad C_{-1}^{(\nu)} = \frac{A^\nu - 1}{\nu} - \int_0^A \frac{G(t)}{t} dt$$

in equation (5.9).

The number A has the same meaning here as in the cited work.

Obviously, this method (which is convenient to use when the function $G(t)$ converges rapidly to t^ν as $t \to \infty$) is not the only one for obtaining an approximation of (5.5).

We note also that in some instances it may be advantageous not to expand the continuous part of the kernels (5.2) into a series of polynomials of the type (5.6), but to express it as a double series of Jacobi polynomials (or in special cases as series in Gegenbauer, Legendre or Chebyshev polynomials).

6. The expansions obtained in Section 3 are useful for solving certai integral equations of the second kind. We shall show this by means of [14]

$$\varphi(\xi) + \frac{c}{\pi}\int_{-1}^{1} \ln \frac{1}{2|\tau - \xi|}\, \varphi(\tau)\, d\tau = f(\xi), \qquad -1 < \xi < 1 \tag{6.1}$$

Denoting the solution of equation (6.1), when $f(\xi) \equiv 1$, by $\chi(\xi)$, we substitute into it the expansion for the logarithm (Section 3). By termwise integration we obtain

$$\chi(\xi) = 1 - \frac{2c}{\pi}\sum_{m=1}^{\infty} X_m T(\xi) \qquad \left(X_m = \frac{1}{m}\int_{-1}^{1} \chi(\tau)\, T_m(\tau)\, d\tau\right) \tag{6.2}$$

Multiplication of both sides of the last equation by $T_n(\xi)$, and integration, yield the following infinite system

$$n X_n = A_{n0} - \frac{2c}{\pi}\sum_{m=1}^{\infty} A_{nm} X_m \qquad (n = 1, 2, 3, \ldots \infty) \tag{6.3}$$

$$A_{nm} = 0 \quad (n + m = 1, 3, 5, \ldots).$$

$$A_{nm} = [1 - (m+n)^2]^{-1} + [1 - (m-n)^2]^{-1} \quad (n + m = 2, 4, 6, \ldots)$$

Having considered an even system (6.3), i.e. the case when $n = 2q$, and an odd system when $n = 2q - 1$, one can show that

$$\sum_{m=1}^{\infty} A_{nm} = \frac{1}{4q^2 - 1} \qquad (q = 1, 2, 3, \ldots \infty)$$

Taking into account this last equation, we can easily obtain the condition for the regularity of the infinite system (6.3) in the form $c < 15\pi$. Having found the approximate solution of system (6.3) (for example by terminating these expansions at a certain place) we have obtained an approximate solution also for the integral equation (6.1) with $f(\xi) \equiv 1$. The solution of the integral equation with any right-hand side is not difficult to find now with the aid of formulas given in [11].

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